

Ordered and Periodic Chaos of the Bounded One-Dimensional Multibarrier Potential

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Numerical analysis indicates that there exists an unexpected new ordered chaos for the bounded one-dimensional multibarrier potential. For certain values of the number of barriers, repeated identical forms (periods) of the wavepackets result upon passing through the multibarrier potential.

KEY WORDS: bounded multibarrier array; chaos; Lanczos tridiagonalization method.

1. INTRODUCTION

Recent studies of the chaotic aspects of different complex systems have resulted in finding certain conditions under which the related chaos become *ordered*, predicted and deterministic. Among these one may include the early ordered chaos found in the one-dimensional logistic maps (Kaneko, 1984) or the optical “chaos itinerancy” (Arecchi *et al.*, 1990; Otsuka, 1990) in which the onset of chaos occurs in a rather ordered manner. Another known example of ordered chaos has been shown in Hondou and Sawada (1995) for the one-dimensional periodic potential with constant slope in which the time series are produced by a tent map (Hondou and Sawada, 1995). The particle which passes through such a potential is shown (Hondou and Sawada, 1995) to be under the influence of an ordered chaos which is effected in an unexpected drift in the opposite direction to that of the average potential gradient (Hondou and Sawada, 1995). It is also shown (Hondou and Sawada, 1995) for the case of zero-average gradient that the smaller become the widths of the potential barriers the more increased is the transition probability and the corresponding ordered chaos (Hondou and Sawada, 1995).

Similar results were shown (Bar and Horwitz, 2002) regarding the bounded one-dimensional multibarrier potential which certainly has no average gradient and in which the width of its barriers is inversely proportional to the number of them.

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It is found in Bar and Horwitz (2002), analogously to Hondou and Sawada (1995), that the larger is the number of barriers along the same spatial length, which means that the smaller are their width, the higher becomes the transition probability of this system. Thus, since the bounded multibarrier system was shown in Bar and Horwitz (2002) to be chaotic then, according to the criterion in Hondou and Sawada (1995) for ordered chaos, we conclude that this system belongs to the last class.

Although quantum chaos is generally expected to exist in systems in which the classical limit is chaotic (Lewenkopf, 1990), there are, nevertheless, quantum systems which show chaotic signs without classical counterparts. This has been shown in Ashkenazi *et al.* (1995) for the quantum one-dimensional single barrier and in Bar and Horwitz (2002) for the bounded one-dimensional multibarrier potential. Also, the appearance of chaotic behavior in these one-dimensional systems in Bar and Horwitz (2002) and Ashkenazi *et al.* (1995) is due to their being bounded and composed of a not large number of barriers. Thus, in the limit of a very large number of barriers arrayed along the whole axis as, for example, the one-dimensional Kronig–Penney system (Merzbacher, 1961), one should not expect chaotic effects (see also the discussion in Wobst *et al.* (2003) of the localization–delocalization transition).

Our main goal in this work is to show that changing the number and/or the width of the barriers results in the appearance of *new* ordered chaos which is effected in the form of periods. These are neither periods in time nor in space but periods in the number of barriers N . We note that by the phrase new ordered chaos we do not mean that the observed wavepackets become less chaotic and complex but that the same chaotic structure is seems to be repeatedly observed as the number of barriers N increases by specific values.

We note, as mentioned, that chaotic-like effects were discussed (Ashkenazi *et al.*, 1995) with respect to *one* rectangular barrier and it was found that the chaotic appearance of the passing wavepacket as well as its correlation with the initial one critically depend upon the width of the single barrier. Thus, it seems appropriate to extend the discussion to the multibarrier potential and find the conditions under which the related chaotic effects become periodically ordered.

We use in our discussion the well-known fact that a classical (or semiclassical) wavepacket spreads and becomes chaotic (Ashkenazi *et al.*, 1995; Bar and Horwitz, 2002; Brody *et al.*, 1985; Haller *et al.*, 1983; Pattanayak and Schieve, 1994; Reichel, 1992; Zaslavsky, 1981) when it passes through a region along which a system of potential barriers (or wells) is arranged. The degree of the resulting chaos may be determined from the correlation (Ben-Avraham and Havlin, 2000) between the initial and final forms of the passing particle (wavepacket). Thus, if this correlation turns out to be small then the initial wavepacket has been considerably changed and its chaotic effects have increased in the passage through the barriers. In the following we use the Lanczos tridiagonalization method (Cullum and Willoughby, 1980, 1981, 1985; Lanczos, 1950; Marko, 1995) for calculating

the correlation (and, therefore, the degree of chaos) between the initial wavepacket and the one which emerges from the potential array.

We show, using the obtained data of the correlation, that the passing chaotic wavepackets are, under certain values of N and c (which is the ratio of total interval to total width of the potential array), strictly periodic and predicted. That is, suppose that a specific wavepacket which pass through a multibarrier potential with a given N , c and total length L assumes some specific form. Then it is shown that the same initial wavepacket assumes exactly the same form with the corresponding correlation when it pass through $(N + nP)$ barriers arranged along the same length L and c where n is the positive numbers $1, 2, 3, \dots$ and P are the periods.

We also show that although the passing wavepackets are highly sensitive to variations of the ratio c , there are, nevertheless, some specific values of N for which the waveforms and the corresponding correlations are preserved for whole ranges of c .

In Section 2, we present the bounded one-dimensional multibarrier potential and the formalism of the energy level statistics (Bar and Horwitz, 2002; Brody *et al.*, 1985; Haller *et al.*, 1983; Reichel, 1992) used for discussing its chaotic properties. This method were used in Bar and Horwitz (2002) for demonstrating that this system is chaotic. We use here numerical analysis for further discussing the mentioned periodic ordered aspect of these chaotic effects. We show the dependence of the passing wavepackets (and their corresponding correlations with the given initial one) upon the number of barriers N and the ratio c . In Section 3, we demonstrate the mentioned order which is effected: (1) through the remarked periods P which turns out to be of two kinds; one is very frequent and is effective for a large specific values of N and the other is rare and shows up only in two specific values of N (from the range $2 \leq N \leq 72$) and (2) through the constancy of the passing wavepackets and their related correlations for certain ranges of the ratio c . In Table I we show the correlations for some specific values of N and for 6 different values of the ratio c and indicate the correlations which are periodic. We conclude in Section 4 with a summary of the main points.

2. THE CORRELATION BETWEEN THE INITIAL AND FINAL WAVEPACKETS FOR THE BOUNDED ONE-DIMENSIONAL MULTIBARRIER POTENTIAL

The bounded one-dimensional multibarrier potential discussed here is supposed to be arranged along the x axis between the points $x = -10$ and $x = 10$. Assuming that the number of barriers in the system is N one may introduce (Bar and Horwitz, 2002) the variables a and b which respectively denote the total width of the N barriers, where the potential V satisfies $V > 0$, and the total interval among them where $V = 0$. Thus, one may realize (Bar and Horwitz, 2002) that the width of each barrier is $\frac{a}{N}$ and the interval between any two neighboring ones

is $\frac{b}{(N-1)}$. Denoting the ratio of b to a by c and the total length of the system $a + b$ by L one may express (Bar and Horwitz, 2002) a and b in terms of c and L as

$$a = \frac{L}{1+c} \quad b = \frac{Lc}{1+c} \quad (1)$$

The possible existence of chaotic properties for any bounded one-dimensional multibarrier (or multiwell) potential system is usually determined by applying the energy level statistics (Bar and Horwitz, 2002; Brody *et al.*, 1985; Haller *et al.*, 1983; Reichel, 1992). In this method one begins from the following two-dimensional matrix equation

$$\begin{bmatrix} A_{2N+1} \\ B_0 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A_0 \\ B_{2N+1} \end{bmatrix}, \quad (2)$$

where A_{2N+1} and B_{2N+1} are the amplitudes of the transmitted and reflected parts respectively of the passing wavapacket from the N th potential barrier. A_0 is the transmission coefficient of the initial wave that approach the first barrier and B_0 is the reflected part from this barrier. The components S_{11} , S_{12} , S_{21} , and S_{22} are the matrix elements of the two-dimensional S matrix which are related to the corresponding transfer matrix Q of the multibarrier potential (see, for example, equation (21) in Bar and Horwitz, 2002). The energy level statistics method (Brody *et al.*, 1985; Haller *et al.*, 1983; Reichel, 1992) is used by imposing boundary value conditions at the remote boundaries of the system. In Bar and Horwitz (2002) periodic boundary conditions are used at the points $|x| = R$, where R is much larger than the total length $L = a + b$ of the system, so that one obtains

$$\begin{aligned} A_{2N+1}f(R) &= A_0f(-R) \\ B_{2N+1}f(-R) &= B_0f(R), \end{aligned} \quad (3)$$

where $f(R)$ and $f(-R)$ denote the wavepackets at the points $x = R$ and $x = -R$ respectively. Using Eq. (3) one may write Eq. (2) as

$$\begin{bmatrix} A_{2N+1} \\ B_0 \end{bmatrix} = \frac{f(R)}{f(-R)} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A_{2N+1} \\ B_0 \end{bmatrix} \quad (4)$$

In order to obtain a nontrivial solution for the vector $\begin{bmatrix} A_{2N+1} \\ B_0 \end{bmatrix}$ we must solve the following equation

$$\det \begin{bmatrix} \frac{f(R)}{f(-R)}S_{11} - 1 & \frac{f(R)}{f(-R)}S_{12} \\ \frac{f(R)}{f(-R)}S_{21} & \frac{f(R)}{f(-R)}S_{22} - 1 \end{bmatrix} = 0 \quad (5)$$

In Bar and Horwitz (2002) we have used for $f(R)$ ($f(-R)$) the plane wave e^{ikR} (e^{-ikR}) and have expressed the S matrix elements in terms of the known transfer matrix elements (see equations (15) and (21) in Bar and Horwitz, 2002). As a result of these substitutions one obtains from Eq. (5) a complex equation from which the appropriate energies which correspond to its real and imaginary parts are derived (Bar and Horwitz, 2002). Figure (8) in Bar and Horwitz (2002) shows the level spacing distribution of these energies in the form of a histogram which is clearly of the chaotic Wigner type (Brody *et al.*, 1985; Haller *et al.*, 1983; Reichel, 1992).

In this work we use, instead of plane waves, a semiclassical complex Gaussian wavepacket since this kind of wave function tends easily to be deformed and becomes chaotic upon passing a multibarrier (or multiwell) potential (Bar and Horwitz, 2002). Also, the semiclassical character of the wavepacket enables one to simultaneously discuss, as done in the following, its momentum and position. Note that even in the quantum regime one may introduce the coherent state formalism (Glauber, 1963; Swanson, 1992) which allows one (Swanson, 1992) to simultaneously define the expectation values of the conjugate variables Q and P . We note that in the numerical part of this work all the wavepackets (denoted ϕ), including the initial one, are numerically and graphically constructed from a given complex Gaussian packet P_{packet} given by

$$P_{\text{packet}}(x, t, x_0, p_0, w_0) = \frac{\sqrt{w_0\pi}^{\frac{1}{4}} e^{-\frac{p_0^2}{4w_0^2}} e^{\frac{w_0^2 \left(i(x_0-x) - \frac{p_0}{2w_0} \right)^2}{1-2itw_0^2}}}{\sqrt{1-2itw_0^2}}, \tag{6}$$

where x_0 is the initial value of the mean position of the packet in coordinate space and p_0 and w_0 are the initial momentum and width in p space. The width w_0 is, actually, the initial uncertainty in the momentum. For an effective numerical simulation the space and time variables were discretized (Bar and Horwitz, 2002; Marko, 1995) with a resolution of $dx = \frac{1}{7}$ and $dt = \frac{1}{50}$ so that we obtain $dx^2 > dt$ which is necessary for stabilizing and steadying the relevant numerical method (Bar and Horwitz, 2002; Marko, 1995). For the initial x_0 , p_0 and w_0 we choose the values of $x_0 = -10$, $p_0 = 3$ and $w_0 = \frac{1}{2}$. In the semiclassical discussion adopted here we assume that the wavepacket is associated with a particle of mass m where for m we assign the value of $\frac{1}{2}$. Thus, as in (Bar and Horwitz, 2002), the units we use for x , t and p are $x = \frac{x_{\text{cm}}}{h}$, $t = \frac{t_{\text{sec}}}{mh}$ and $p = mv$. That is, one may realize in this scaling that the velocities in $\frac{\text{cm}}{\text{sec}}$ are related to the mentioned parameters x and t by $\frac{x_{\text{cm}}}{t_{\text{cm}}} = \frac{x}{mt}$. In order to maintain the condition of $E > V$, where E is the energy of the passing wavepacket, we assign for the constant height of the barriers the value of $V = 2$.

The initial wavepacket which approach the multibarrier potential is expressed as

$$\phi(t = 0) = Re^2(P_{\text{packet}}) + Im^2(P_{\text{packet}}), \tag{7}$$

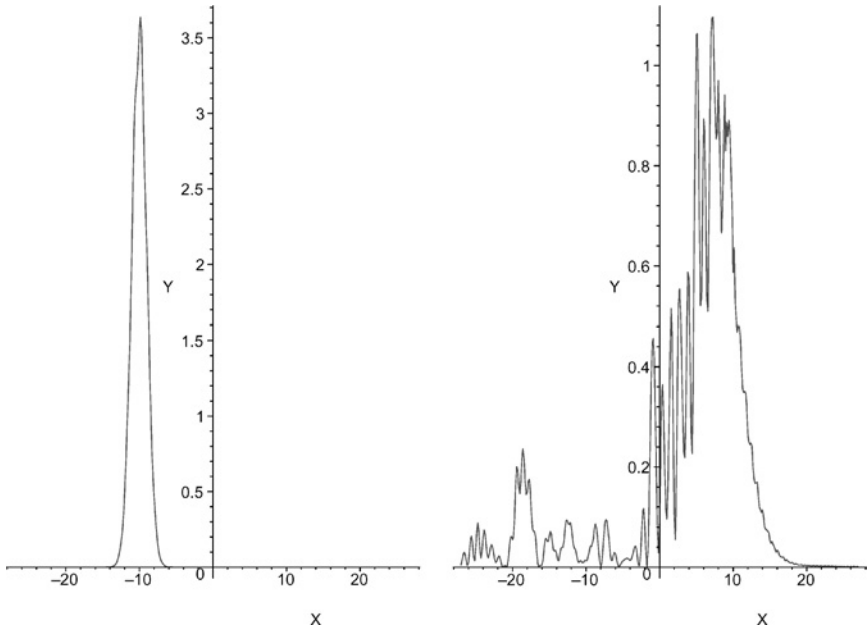


Fig. 1. The left-hand side figure shows the initial wavepacket from Eq. (7) as function of x where x is given in units of $\frac{x_{cm}}{h}$. The right-hand side graph shows this wavepacket at time $t = 6$ after passing a multibarrier potential composed of 10 barriers whose ratio of total interval to total width is $c = 1$. Note how expanded and deformed the wavepacket becomes. Also note that the multibarrier potentials are not shown in this figure and in Fig. 2.

where $Re(P_{packet})$ and $Im(P_{packet})$ denote the real and imaginary parts respectively of P_{packet} from Eq. (6). The initial wavepacket of Eq. (7) is shown at the left-hand side panel of Fig. 1. We note that by its definition the initial wavepacket $\phi(t = 0)$ from Eq. (7) spreads with time without having to pass through any potential (see Eq. (6)). We are not interested here in this kind of known spreading but, especially, want to track and follow the unknown chaotic-like deformation of the packet due to its passage through the multibarrier potential. Thus, we numerically follow the time evolution of the real and imaginary parts through the multibarrier potential and obtain the passing wavepacket as

$$\phi(t) = Re^2(P_{packet, v}) + Im^2(P_{packet, v}), \tag{8}$$

where $Re(P_{packet, v})$ and $Im(P_{packet, v})$ denote the real and imaginary parts of P_{packet} after passing through the multibarrier potential.

We note that the number of the different chaotic wavepackets which evolve from the initial wavepacket of Eq. (7) is very large. For example, by slightly changing the ratio c or by adding or removing even one barrier results in a

completely different wavepacket compared to the one which corresponds to the potential before the change. Since the passing wavepacket becomes chaotic there is generally no rule that controls its form or the correlation C between it and the initial wavepacket from Eq. (7). As remarked, we show in the following section that for certain values of N and c one may predict the forms, and therefore the related correlation, of the passing wavepackets.

The right-hand side panel of Fig. 1 shows how the initial wavepacket from the left-hand side expands and become deformed at time $t = 6$ after passing through a ten-barrier potential whose ratio c is unity. The influence of changing N upon the passing wavepacket and its corresponding correlation C is further demonstrated at the left-hand side panel of Fig. 2 which shows the waveform obtained for the same c and t as those of the right-hand side panel of Fig. 1 but for a 15-barrier potential. Note that by increasing N by 5 the wave packet becomes more chaotic and deformed compared to that at the right-hand side of Fig. 1. The dependence

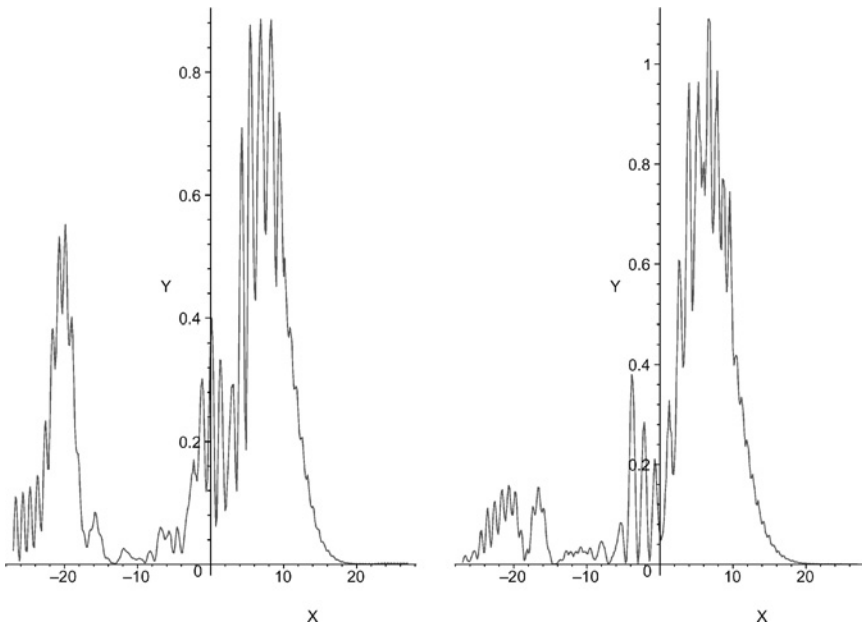


Fig. 2. The left-hand side panel shows how the initial wavepacket from Eq. (7) changes at time $t = 6$ after passing a multibarrier potential composed of 15 barriers whose ratio is $c = 1$. Comparing this graph to the right-hand side panel of Fig. 1 one may realize that the wavepacket becomes more deformed and chaotic by increasing N from 10 to 15 retaining the same ratio of $c = 1$. The right-hand side panel shows the wavepacket obtained at time $t = 6$ for the same 15 barrier potential as that of the left-hand side panel but with a ratio of $c = \frac{1}{9}$. The decrease in c results in a wavepacket which is less complex compared to that at the left-hand side.

of the passing wavepacket upon c is shown at the right-hand side panel of Fig. 2 which is drawn for the same N and t as those of the left-hand side but for $c = \frac{1}{9}$. That is, decreasing c from unity to $\frac{1}{9}$ causes the passing wavepacket to become much less complex and chaotic compared to the form at the left-hand side.

We refer in the following to Table I which shows the correlation between the same initial wavepacket of Eq. (7) and the passing one for 40 values of N and for six different values of c : $c = 4, \frac{7}{3}, \frac{3}{2}, 1, \frac{2}{3}, 0.25$. The spatial length of the multibarrier was fixed to $L = 20$ and the time at which all the passing wavepackets were calculated is $t = 6$ which corresponds to increasing 300 times the mentioned time interval of $dt = \frac{1}{50}$. Thus, at this time the initial wavepacket have passed through all the barriers arranged along the fixed length of L . Each of the tabulated values of the correlation was numerically calculated using the Lanczos tridiagonalization method (Cullum and Willoughby, 1980, 1981, 1985; Lanczos, 1950; Marko, 1995) which yields a tridiagonal matrix the values in its principal diagonal are the sought-for correlations. In this method the better and accurate result is given by the matrix element located at the bottom of the principal diagonal. Thus, the larger is the tridiagonal matrix the more accurate becomes the correlation associated with this matrix element. This is due to the large number of numerically running the program which generally yields better results. We note, however, that the exact values of the correlation is not of our main concern here but especially we concentrate our attention upon its dependence on c and N . These dependencies may be established even from a small tridiagonal matrix if we use consistently the same order of it for all c and N . Thus, we use for all the numerical work here a third order Lanczos tridiagonalization matrix. An example of such a matrix is the following one which corresponds to $c = \frac{7}{3}$, $N = 10$, $L = 20$, $x_0 = -10$, $p_0 = 3$, $w_0 = \frac{1}{2}$, $t = 6$ and $V = 2$.

$$M(N = 10, c = 7/3) = \begin{bmatrix} 2.630821735 & 33.07455189 & 0 \\ 33.07455189 & 1396.833502 & 52187.77697 \\ 0 & 52187.77697 & 1669.69722 \end{bmatrix}$$

The values of the correlation between the passing wavepacket at the time $t = 6$ and the initial one from Eq. (7) are tabulated along the principal diagonal. The value of 1669.69722 at the bottom of this diagonal is, as remarked, more accurate than the two other values. The tridiagonal matrix is symmetric which means that the off diagonal matrix elements which are symmetrically located about the principal diagonal are equal. These off diagonal elements are the normalizing factors of the diagonal elements (Cullum and Willoughby, 1980, 1981, 1985; Marko, 1995).

Each one of the values tabulated in Table I is obtained from the bottom value of the principal diagonal of the corresponding tridiagonal matrix. The 40 rows in Table I correspond to the three representative ranges of $N = 4, \dots, 15$, $N = 31, \dots, 40$ and $N = 55, \dots, 72$ and the six columns to the six values of the

ratio $c = 4, \frac{7}{3}, \frac{3}{2}, 1, \frac{2}{3}, 0.25$. As seen from the table the correlation ranges, for the specific values given here to the related parameters L, w_0, x_0, p_0, V and t , over values which greatly differ among them. Thus, in order to be able to graphically plot the correlation as function of N (or c) one have to scale the ordinate axis in a log basis as done in Figs. 3 and 4. We must remark that Figs. 3–5 are constructed not only from the tabulated values of Table I but also from other values which are not given in this table. That is, the correlation values used for Figs. 3–5 are for all N from $N = 4 + n, n = 1, 2, \dots, 68$ and for $c = 4, \frac{7}{3}, \frac{3}{2}, 1, \frac{2}{3}, 0.25$. From Table I and Figs. 3 and 4 one may realize that the correlation changes in a stochastic and unexpected manner even when adding or removing only one barrier. Also, one may see that the larger values of the correlation C are found at either large or small values of the ratio c and the smaller values of C are found at the intermediate values of c . This is shown in Fig. 3 in which we compare at the left-hand side panel of it the correlation C as function of N for $c = 4$ (continuous curve), $c = \frac{7}{3}$ (dashed curve) and $c = \frac{3}{2}$ (dashdot curve). At the right-hand side panel of Fig. 3 we compare the correlation C , as function of N , for $c = 4$ (continuous curve), $c = 1$ (dashed curve) and $c = \frac{2}{3}$ (dashdot curve). Remembering that the ordinate is scaled in a log basis one may realize, for example, how large is the difference for $10 \leq N \leq 25$ between the correlation for $c = 4$ and those obtained for $c = \frac{3}{2}, c = \frac{2}{3}$ and $c = 1$. Similar differences are demonstrated at the left-hand side panel of Fig. 4 where the correlation C , as function of N , for $c = 0.25$ (continuous curve) is compared to those for $c = \frac{7}{3}$ (dashed curve) and $c = \frac{3}{2}$ (dashdot curve). Note again the large differences for $12 \leq N \leq 25$ between the correlation C obtained for $c = \frac{3}{2}$ and those for $c = 0.25$ and $c = \frac{7}{3}$. The similarity of the correlations for large and small c is demonstrated at the right-hand side panel of Fig. 4 where we compare C for $c = 4$ (continuous curve) to that for $c = 0.25$ (dashed curve). Note, however, the large difference between these correlations for $28 \leq N \leq 36$ where the C 's for $c = 4$ are much larger compared to those for $c = 0.25$.

From Table I and the corresponding panels of Figs. 3 and 4 one finds the appropriate N and c for constructing a bounded one-dimensional multibarrier potential from which one may obtain a large correlation between the initial and final wavepackets.

3. THE PERIODS OF CHAOS

As realized from the former section the wavepacket which passes through the multibarrier potential becomes deformed and chaotic and the emerging waveforms are quite different even for neighboring values of N and c . That is, the resulting waveforms $\phi(t)$ and the corresponding correlations C depend upon c and N in such a manner that slightly changing either one of them results in a large change of $\phi(t)$ and C . We have, nevertheless, found that there exists an unexpected order among the multitude of the chaotic waveforms and the corresponding correlations. This

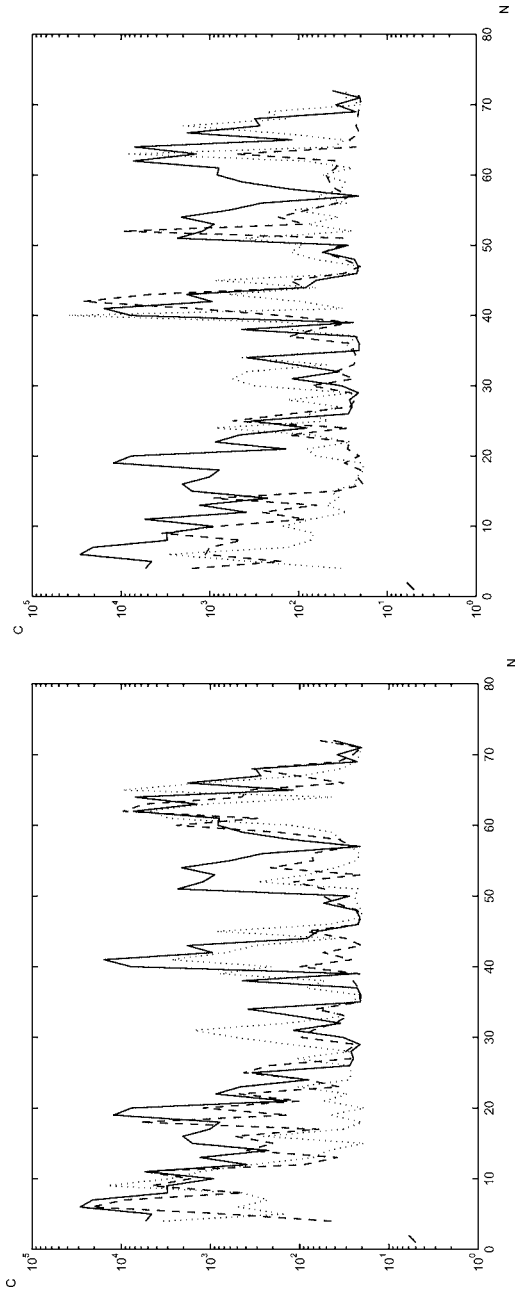


Fig. 3. At the left-hand side panel we compare the correlation C as function of N for $c = 4$ (continuous curve) to those for $c = \frac{7}{3}$ (dashed curve) and $c = \frac{3}{2}$ (dashdot curve). The right-hand side panel compare the correlation C for $c = 4$ (continuous curve) to those for $c = 1$ (dashed curve) and $c = \frac{3}{2}$ (dashdot curve). Remembering that the ordinate axis is scaled in a log basis one may realize that the correlations for $c = 4, c = \frac{7}{3}, c = \frac{3}{2}, c = \frac{3}{2}$ and $c = 1$ widely differ from each other. See, especially, the differences, for $10 \leq N \leq 25$, among $c = 4, c = \frac{3}{2}, c = \frac{3}{2}$ and $c = 1$.

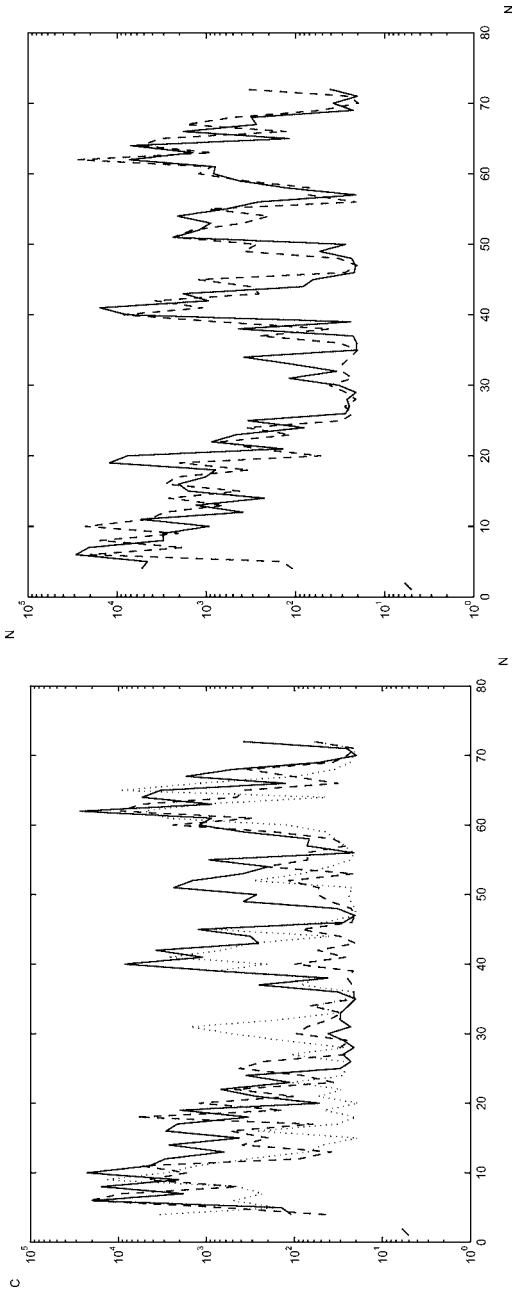


Fig. 4. At the left-hand side panel we compare the correlation C as function of N for $c = 0.25$ (continuous curve) to those for $c = \frac{2}{3}$ (dashed curve) and $c = \frac{3}{7}$ (dashdot curve). Taking into account that the ordinate axis is scaled in a log basis one may realize, for example, the large differences for $12 \leq N \leq 25$ between the correlations C for $c = \frac{2}{3}$ and those for $c = 0.25$ and $c = \frac{3}{7}$. At the right-hand side panel we compare the correlations C for $c = 4$ (continuous curve) to those for $c = 0.25$ (dashed curve). From the similarity between the two graphs (which, however, greatly differ for $28 \leq N \leq 36$) one may see that the correlations are large for either large or small c .

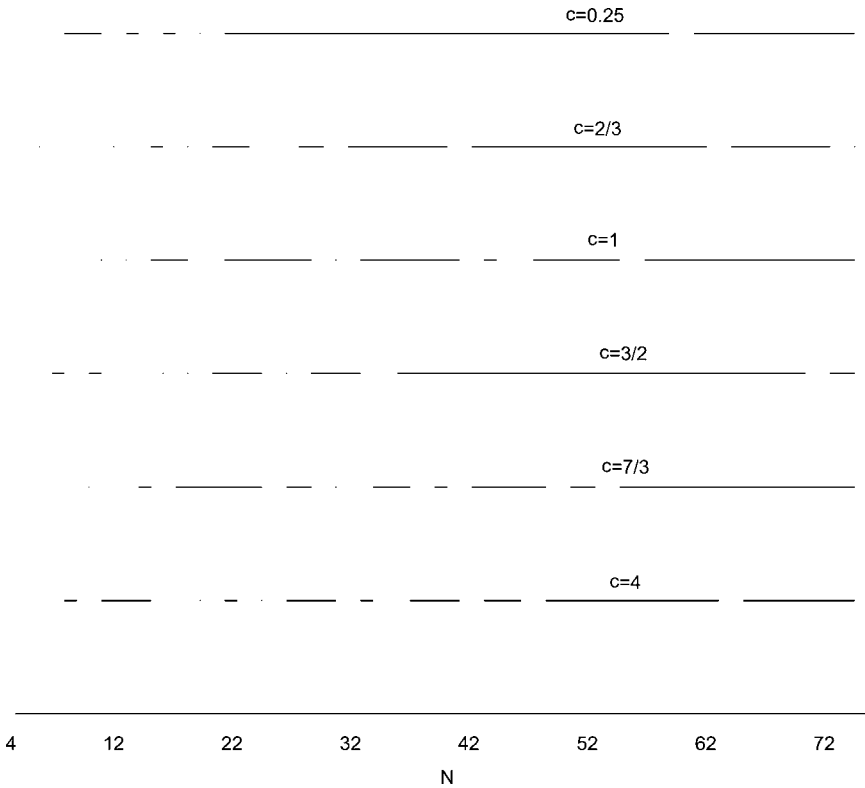


Fig. 5. From this figure one may compare and find if the correlations C , for the six values of $c = 0.25, \frac{2}{3}, 1, \frac{3}{2}, \frac{7}{3}, 4$, have the large period of $P = 140$. The six values of c label the 6 horizontal lines. A periodic C , with given c and N , is denoted by a point and the absence of this point signifies a nonperiodic C . Thus, as seen from the figure the larger is N in the abscissa the longer become the linear continuous sections for all c which denote that the number of periodic C becomes large. The wide gaps at these lines for small N signify that there exists a large number of nonperiodic C at these values of N .

order is reflected in periodicities which are clearly observed for specific values of N and c . That is, we find that exactly the same identical wavepackets emerge from the potential barriers when the number of the latter increases by specific numbers P or by any integral multiplication of them where the total spatial length of the system and the ratio c remain fixed. The specific periods P are found to be of two kinds: a large period of $P_L = 140$ and a small one of $P_S = 28$. That is, if the relevant N (for a specific c) is periodic then exactly the same wavepacket emerge from all the potentials which have $(N + nP)$ barriers where n denote the whole numbers $1, 2, \dots$ and P is either the large period of 140 or the smaller one of 28.

Note that since $5 \times 28 = 140$ then any N and c which are characterized as being periodic with the small period of 28 are also automatically periodic with the larger period of 140.

The large period of 140 is found to be very frequent and common for a large number of different c and N whereas the smaller period of 28 is rare. The criterion used here for characterizing any pair of N and c as periodic is that they have the same identical tridiagonal matrix for all $(N + nP)$ where $n = 0, 1, 2, \dots$ and P is either 140 or 28. In Table I we have denoted the multibarrier potentials which are periodic with the large period of 140 by the word p attached to the numerical values of the corresponding correlations C . All the other values in Table I in which the word p is absent are nonperiodic. Thus, as seen from the table there exists a large number of periodic multibarrier potentials which produce the same wavepackets and the same correlations when the number of barriers are increased by 140 or by any integral multiplication of it. It is found (see Table I) that the smaller is the number of barriers N the more frequent is the number of the nonperiodic potentials and as N increases the periodicity of the corresponding multibarrier potentials becomes more common and frequent.

In Fig. 5 we have schematically drawn for the six values of c the correlations C as functions of N from the point of view of whether they are periodic with the large period of 140 or not. That is, each periodic C , which corresponds to some specific N and c , is denoted by a point and the absence of this point for some given N and c signifies that the relevant C is nonperiodic. Note that, as remarked, the values of N used in this figure are not only those of Table I but all the values of $N = 4 + n$, $n = 0, 1, 2, \dots, 68$. From the frequent occurrence of the gaps in the horizontal lines of Fig. 5 for small N and from the width of these gaps one may realize that there exists a large number of nonperiodic correlations C at these values of N . The larger N becomes the more rare and narrow these gaps become which means that the number of the periodic correlations C increases for all values of c . For very high values of N (not shown in Fig. 5) the continuous linear sections become very long for all c which means that all the wavepackets as well as their corresponding correlations are periodic with the large period of 140.

Regarding the smaller period of 28 we have found that it exists for the two pairs of $(c = 4, N = 29)$ and $(c = 0.25, N = 28)$. That is, the same identical tridiagonal matrix, which implies the same emerging wavepacket and correlation, is obtained for the $c = 4$ case for all potentials (arranged along the same fixed length of $L = 20$) which have $(29 + n \times 28)$ barriers where $n = 0, 1, 2, \dots$. Likewise, for the $c = 0.25$ case one obtains the same tridiagonal matrix, wavepacket and correlation (which are not the same as those of the formerly discussed $c = 4$ case) for all potentials which have $(28 + n \times 28)$ barriers arrayed along the same fixed length of $L = 20$ where $n = 0, 1, 2, \dots$.

Another kind of ordered regularity which we have found among the multitude of all the chaotic wavepackets is related to some specific tridiagonal matrices (and,

therefore, to their corresponding wavepackets and correlations) which remain constant even when the ratio c changes. That is, although the general behavior is the unexpected change and deformation of the passing wavepacket when c changes even slightly there exist, nevertheless, specific values of N which are characterized as related to correlations and wavepackets which retain their forms even when c is changed. For example, for $N = 70$ we have found that the same tridiagonal matrix (which means the same wavepacket and correlation) remains constant for $1 \geq c \geq 0.25$. The same situation is also encountered for $N = 71$ where this time the constancy of the matrix, wavepacket and correlation are retained for the larger range of $4 \geq c \geq 1$. The relevant tridiagonal matrix for the last case is

$$M(N = 71, 1 \leq c \leq 4) = \begin{bmatrix} 0.06426037493 & 6.899198452 & 0 \\ 6.899198452 & 68.29688359 & 562.8969638 \\ 0 & 562.8969638 & 20.28518401 \end{bmatrix}$$

The same situation is again encountered for other values of N for which one finds constant different matrices. These N 's and the corresponding ranges of c over which the emerging wavepackets (and the appropriate correlations C) retain their forms are; at $N = 47$ for $\frac{3}{2} \geq c \geq \frac{2}{3}$, at $N = 24$ for $4 \geq c \geq \frac{7}{3}$ and at $N = 18$, $N = 24$, $N = 53$, and $N = 123$ for $1 \geq c \geq \frac{2}{3}$. Note that all these values of N are also characterized as being periodic with the large period of 140. Thus, one may realize that the constancy of the relevant tridiagonal matrices (and the corresponding wavepackets and correlations) are retained not only for these specific N but also for all the other N 's obtained by increasing them by 140 or by any integral multiplication of it. In other words, this behavior of constant waveforms (and correlations) when c changes is strongly related to the previously mentioned behavior of periodic waveforms (and correlations) when the number of barriers, for these specific N and c , increases by 140 or by any integral multiplication of it.

4. CONCLUDING REMARKS

We have discussed the chaotic deformed wavepackets which come out of a bounded one-dimensional multibarrier potential and study the correlation of these wavepackets with the initial one. It has been shown, using the Lanczos tridiagonalization method, that there exists an unexpected order and regularity among the multitude of all the possible chaotic wavepackets which come out of this system. This order is characterized by the existence of two periods through which one may obtain the same wavepackets and correlations when the number of barriers increase, for some specific N and c , by either 140 or 28 or by any integral multiplication of them. The more common and frequent period is that of 140 whereas the smaller one of 28 is rare. Any wavepacket and its corresponding

Table I. Correlations C Between the Passing Wavepacket at Time $t = 6$ and the Initial One From Eq. (7)

N	Correlation C for $(c = 4)$	Correlations C for $(c = \frac{7}{3})$	Correlations C for $(c = \frac{3}{2})$	Correlations C for $(c = 1)$	Correlations C for $(c = \frac{2}{3})$	Correlations C for $(c = 0.25)$
4	5.31202×10^3	4.392×10^1	3.36365×10^3	1.5939×10^3	3.313×10^1	1.0855×10^2
5	4.55098×10^3	6.380×10^2	1.5324×10^2	1.5478×10^2	3.4617×10^2	1.4041×10^2
6	2.89038×10^4	2.143528×10^4	4.8818×10^2	1.12423×10^3	$2.97818 \times 10^3 p$	1.995463×10^4
7	2.076735×10^4	9.20498×10^3	$2.2882 \times 10^2 p$	1.00467×10^3	1.3024×10^2	1.82352×10^3
8	$3.01606 \times 10^3 p$	4.3025×10^2	$4.2984 \times 10^2 p$	4.3489×10^2	7.312×10^1	$1.555678 \times 10^4 p$
9	$3.06900 \times 10^3 p$	4.59357×10^3	1.421014×10^4	3.47268×10^3	6.664×10^1	$2.05691 \times 10^3 p$
10	9.28730×10^2	$1.6697 \times 10^3 p$	$2.88651 \times 10^3 p$	8.2338×10^2	1.4825×10^2	$2.273285 \times 10^4 p$
11	$5.37380 \times 10^3 p$	5.43254×10^3	$1.29778 \times 10^3 p$	$8.378 \times 10^1 p$	7.195×10^1	$4.4193 \times 10^3 p$
12	$3.8843 \times 10^2 p$	8.818×10^1	3.316×10^2	2.5926×10^2	$2.829 \times 10^1 p$	2.93109×10^3
13	$1.29587 \times 10^3 p$	3.8130×10^1	7.086×10^1	6.284×10^1	4.819×10^1	$6.2898 \times 10^2 p$
14	$2.2356 \times 10^2 p$	$3.903 \times 10^2 p$	4.743×10^1	9.1362×10^2	3.501×10^1	$2.63225 \times 10^3 p$
15	$1.58547 \times 10^3 p$	$1.9687 \times 10^2 p$	1.905×10^1	$3.664 \times 10^1 p$	$4.278 \times 10^1 p$	4.228×10^2
31	1.1767×10^2	7.134×10^1	$1.52509 \times 10^3 p$	2.368×10^1	$5.5717 \times 10^2 p$	$2.305 \times 10^1 p$
32	$3.4680 \times 10^1 p$	4.026×10^1	$1.4336 \times 10^2 p$	$3.854 \times 10^1 p$	$3.7924 \times 10^2 p$	$3.017 \times 10^1 p$
33	$1.08940 \times 10^2 p$	$2.887 \times 10^1 p$	3.133×10^1	$2.39 \times 10^1 p$	$4.805 \times 10^1 p$	$2.981 \times 10^1 p$
34	3.77230×10^2	$6.677 \times 10^1 p$	5.853×10^1	$2.21 \times 10^1 p$	$4.3716 \times 10^2 p$	$2.427 \times 10^1 p$
35	2.08800×10^1	$2.086 \times 10^1 p$	$2.562 \times 10^1 p$	$2.402 \times 10^1 p$	$2.46 \times 10^1 p$	$2.021 \times 10^1 p$
36	$2.06600 \times 10^1 p$	$3.886 \times 10^1 p$	$2.12 \times 10^1 p$	$2.522 \times 10^1 p$	$2.991 \times 10^1 p$	$3.214 \times 10^1 p$
37	$2.27400 \times 10^1 p$	2.195×10^1	$8.573 \times 10^1 p$	$1.2641 \times 10^2 p$	$2.11 \times 10^1 p$	$2.5054 \times 10^2 p$
38	$4.34410 \times 10^2 p$	$2.518 \times 10^1 p$	$5.223 \times 10^1 p$	$6.78 \times 10^1 p$	$1.1804 \times 10^2 p$	$4.173 \times 10^1 p$
39	$2.4010 \times 10^1 p$	$2.139 \times 10^1 p$	$7.9903 \times 10^2 p$	$2.556 \times 10^1 p$	$1.1795 \times 10^2 p$	$7.3825 \times 10^2 p$
40	$7.75907 \times 10^3 p$	9.886×10^1	$2.0648 \times 10^2 p$	$1.5902 \times 10^2 p$	3.951835×10^4	$8.43398 \times 10^3 p$
55	$6.0292 \times 10^2 p$	$7.002 \times 10^1 p$	$2.3000 \times 10^1 p$	$8.085 \times 10^1 p$	$1.153 \times 10^2 p$	$9.3828 \times 10^2 p$
56	$2.6082 \times 10^2 p$	$7.428 \times 10^1 p$	$2.1090 \times 10^1 p$	$3.69 \times 10^1 p$	$2.865 \times 10^1 p$	$2.108 \times 10^1 p$
57	$2.109 \times 10^1 p$	$2.591 \times 10^1 p$	$2.6780 \times 10^1 p$	$2.673 \times 10^1 p$	$6.079 \times 10^1 p$	$7.158 \times 10^1 p$
58	$1.2728 \times 10^2 p$	$3.629 \times 10^1 p$	$3.4680 \times 10^1 p$	$3.811 \times 10^1 p$	$3.308 \times 10^1 p$	6.796×10^1
59	$4.305 \times 10^2 p$	$1.6849 \times 10^2 p$	$4.3350 \times 10^1 p$	$4.462 \times 10^1 p$	$3.005 \times 10^1 p$	$3.7354 \times 10^2 p$
60	$8.2816 \times 10^2 p$	$2.38018 \times 10^3 p$	$1.2109 \times 10^2 p$	$4.874 \times 10^1 p$	$5.301 \times 10^1 p$	$1.20593 \times 10^3 p$
61	$7.9283 \times 10^2 p$	$2.966 \times 10^2 p$	$2.84059 \times 10^3 p$	$3.805 \times 10^1 p$	2.472×10^1	$8.6802 \times 10^2 p$
62	7.30202×10^3	$9.50662 \times 10^3 p$	$4.9941 \times 10^3 p$	$3.553 \times 10^1 p$	$8.411 \times 10^1 p$	$2.717475 \times 10^4 p$
63	$1.41355 \times 10^3 p$	$5.51001 \times 10^3 p$	$1.15584 \times 10^3 p$	$4.9719 \times 10^2 p$	$8.17489 \times 10^3 p$	$8.6817 \times 10^2 p$
64	$6.98547 \times 10^3 p$	$4.4736 \times 10^2 p$	$4.3380 \times 10^1 p$	$2.241 \times 10^1 p$	$4.017 \times 10^1 p$	$5.33998 \times 10^3 p$
65	$1.187 \times 10^2 p$	$3.811 \times 10^2 p$	$9.71896 \times 10^3 p$	$2.655 \times 10^1 p$	$3.04 \times 10^1 p$	$3.29676 \times 10^3 p$
66	$1.80058 \times 10^3 p$	$3.21 \times 10^1 p$	$1.22609 \times 10^3 p$	$2.057 \times 10^1 p$	$1.8292 \times 10^2 p$	$1.276 \times 10^2 p$
67	$2.7297 \times 10^2 p$	$1.0732 \times 10^2 p$	$9.7360 \times 10^1 p$	$2.244 \times 10^1 p$	$2.01296 \times 10^3 p$	$1.69351 \times 10^3 p$
68	$3.1167 \times 10^2 p$	$3.4067 \times 10^2 p$	$3.4780 \times 10^1 p$	$2.164 \times 10^1 p$	$1.4212 \times 10^2 p$	$5.0016 \times 10^2 p$
69	$2.274 \times 10^1 p$	$3.679 \times 10^1 p$	2.3380×10^1	$2.12 \times 10^1 p$	$2.2945 \times 10^2 p$	$5.241 \times 10^1 p$
70	$3.787 \times 10^1 p$	$2.53 \times 10^1 p$	$2.1300 \times 10^1 p$	$1.983 \times 10^1 p$	$1.983 \times 10^1 p$	$1.983 \times 10^1 p$
71	$2.028 \times 10^1 p$	$2.028 \times 10^1 p$	$2.0280 \times 10^1 p$	$2.028 \times 10^1 p$	2.912×10^1	$2.53 \times 10^1 p$
72	$4.148 \times 10^1 p$	$5.996 \times 10^1 p$	$5.9960 \times 10^1 p$	$2.268 \times 10^1 p$	$2.905 \times 10^1 p$	$3.8058 \times 10^2 p$

Note. The rows correspond to the three ranges of $N = 4 + n, n = 0, 1, \dots, 11, N = 31 + n, n = 0, 1, \dots, 9$, and $N = 55 + n, n = 0, 1, \dots, 17$. The columns correspond to the ratio c for $c = 4, \frac{7}{3}, \frac{3}{2}, 1, \frac{2}{3}, 0.25$. Any N which have correlation C with the word p attached to it is periodic with the large period of 140. That is, any such N have exactly the same value of C for all multibarrier potentials which have $(N + n \times 140)$ barriers where $n = 1, 2, 3, \dots$

correlation which is periodic with the small period is also automatically periodic with the larger one.

The correlation C , as function of either N or (and) c , between the passing wavepackets and the initial one is stochastic and discontinuous as may be realized from Table I and Figs. 3 and 4. One may see from these figures and from Table I that the larger values of C are obtained for $c \approx 4$ or $c \approx 0.25$.

Another ordered behavior that we have found is related to the constancy of the wavepackets and the corresponding correlations for specific N and for whole ranges of c that may be as large as $\Delta c = 3$. All these N 's are also characterized as being periodic with the large period of 140.

In summary, one may see that the chaos demonstrated by the bounded one-dimensional multibarrier potential is an ordered and periodic phenomenon especially for large N .

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